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The Concept of a Noncommutative Riemann Surface

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ABSTRACT

We consider the compactification M(atrix) theory on a Riemann surface Σ of genus $g > 1$. A natural generalization of the case of the torus leads to construct a projective unitary representation of $\pi_1(\Sigma)$, realized on the Hilbert space of square integrable functions on the upper half–plane. A uniquely determined gauge connection, which in turn defines a gauged $\mathrm{sl}_2(\mathbb{R})$ algebra, provides the central extension. This has a geometric interpretation as the gauge length of a geodesic triangle, and corresponds to a 2–cocycle of the 2nd Hochschild cohomology group of the Fuchsian group uniformizing Σ. Our construction can be seen as a suitable double–scaling limit $N \to \infty$, $k \to -\infty$ of a $U(N)$ representation of $\pi_1(\Sigma)$, where k is the degree of the associated holomorphic vector bundle, which can be seen as the higher–genus analog of 't Hooft's clock and shift matrices of QCD. We compare the above mentioned uniqueness of the connection with the one considered in the differential–geometric approach to the Narasimhan–Seshadri theorem provided by Donaldson. We then use our infinite dimensional representation to construct a $C^\star$–algebra which can be interpreted as a noncommutative Riemann surface Σ_θ. Finally, we comment on the extension to higher genus of the concept of Morita equivalence.

1. The quotient conditions.

The $P_- = N/R$ sector of the discrete light–cone quantization of uncompactified M–theory is given by the supersymmetric quantum mechanics of $U(N)$ matrices. In temporal gauge, the action reads

$$S = \frac{1}{2R}\int dt\, \mathrm{Tr}\left(\dot{X}^\mu \dot{X}_\mu + \sum_{\mu>\nu}[X^\mu, X^\nu]^2 + i\Theta^T\dot{\Theta} - \Theta^T\Gamma_\mu[X^\mu,\Theta]\right), \tag{1}$$

where $\mu,\nu = 1,\ldots,9$. The compactification of M(atrix) theory [1]–[3] as a model for M–theory [4] has been studied in [5]. In [6]–[9] it has been treated using noncommutative geometry [10]. These investigations apply to the d–dimensional torus T^d, and have been further dealt with from various viewpoints in [11]–[17]. These structures are also relevant in noncommutative string and gauge theories [18, 19]. Let e_{ij}, $i,j = 1,2$, generate a 2-dimensional lattice in $\mathbb{R}^2$. In compactifying M(atrix) theory on the torus $\mathbb{T}^2$ determined by this lattice one introduces unitary operators $\mathcal{U}_1$ and $\mathcal{U}_2$, defined on the covering space $\mathbb{R}^2$ of $\mathbb{T}^2$, such that

$$\begin{aligned} \mathcal{U}_i^{-1}X_j\mathcal{U}_i &= X_j + 2\pi e_{ij}, \quad i,j=1,2,\\ \mathcal{U}_i^{-1}X_a\mathcal{U}_i &= X_a, \qquad\qquad a = 3,\ldots,9\\ \mathcal{U}_i^{-1}\Theta\mathcal{U}_i &= \Theta. \end{aligned} \tag{2}$$

By consistency the operators $\mathcal{U}_1$ and $\mathcal{U}_2$ commute, up to a constant phase:

$$\mathcal{U}_1\mathcal{U}_2 = e^{2\pi i\theta}\,\mathcal{U}_2\mathcal{U}_1. \tag{3}$$

In this paper we extend Eqs.(2)(3) to the case of compact Riemann surfaces of genus $g > 1$. This is a first step towards the compactification of M(atrix) theory on a Riemann surface. The explicit solutions and their supersymmetry properties will be considered elsewhere.

A Riemann surface Σ of genus $g > 1$ is constructed as the quotient $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the upper half–plane, and $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, $\Gamma \cong \pi_1(\Sigma)$, is a Fuchsian group acting on $\mathbb{H}$ as

$$\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma, \qquad \gamma z = \frac{az+b}{cz+d}. \tag{4}$$

In the absence of elliptic and parabolic generators, the $2g$ Fuchsian generators γ_j satisfy

$$\prod_{j=1}^{g}\left(\gamma_{2j-1}\gamma_{2j}\gamma_{2j-1}^{-1}\gamma_{2j}^{-1}\right) = \mathbb{I}. \tag{5}$$

Inspired by M(atrix) theory, let us promote the complex coordinate $z = x+iy$ to an $N\times N$ complex matrix $Z = X+iY$, with $X = X^\dagger$ and $Y = Y^\dagger$. This would suggest defining fractional

linear transformations of Z through conjugation $\mathcal{U}Z\mathcal{U}^{-1} = (aZ+b\mathbb{I})(cZ+d\mathbb{I})^{-1}$. However, taking the trace we see that this construction cannot be implemented for finite N. Thus we will consider some suitable modification. For the moment note that requiring the $\mathcal{U}_k$ to represent the γ_k gives

$$\prod_{k=1}^{g}\left(\mathcal{U}_{2k-1}\mathcal{U}_{2k}\mathcal{U}_{2k-1}^{-1}\mathcal{U}_{2k}^{-1}\right) = e^{2\pi i\theta}\mathbb{I}, \tag{6}$$

which generalizes the relation of the noncommutative torus (3).

2. The noncommutative torus revisited.

In order to compactify in higher genus it is necessary to extract some general guidelines from the case of the torus. In $g = 1$ the fundamental group is Abelian. This implies that the associated differential generators commute, *i.e.* $[\partial_1, \partial_2] = 0$, so it makes sense to apply the Baker–Campbell–Hausdorff (BCH) formula when computing the phase $e^{2\pi i\theta}$. On the contrary, the fundamental group of negatively curved Riemann surfaces is nonabelian, and the BCH formula is not useful. The derivation of the phase in $g = 1$ by means of techniques alternative to the BCH formula will be the key point to solving the problem in $g > 1$.

Mimicking the case of $g = 1$, one expects the building blocks for the solution to the quotient conditions in $g > 1$ to have the form $e^{\mathcal{L}_n-\mathcal{L}_n^\dagger}$ or $e^{i(\mathcal{L}_n+\mathcal{L}_n^\dagger)}$, for some *gauged* $\mathrm{sl}_2(\mathbb{R})$ *operators* $\mathcal{L}_n$ to be determined. We will show that finding such $\mathcal{L}_n$ is closely connected with the computation of the phase without using the BCH formula. In $g = 1$ the BCH formula is useful, as the commutator between covariant derivatives can be a constant. On the contrary, in $g > 1$, the $\mathcal{L}_n$ will be a sort of gauged $\mathrm{sl}_2(\mathbb{R})$ generators, and $[\mathcal{L}_n, \mathcal{L}_m]$ can never be a c–number.

The solution to the quotient conditions in $g = 1$ is expressed in terms of the exponential of covariant derivatives ∇_k, $k = 1, 2$, so apparently we should use both $e^{\mathcal{L}_n-\mathcal{L}_n^\dagger}$ and $e^{i(\mathcal{L}_n+\mathcal{L}_n^\dagger)}$ when passing to $g > 1$. While the exponential e^{∂_k} will generate translations, the operator $e^{\mathcal{L}_n-\mathcal{L}_n^\dagger}$ will produce $\mathrm{PSL}_2(\mathbb{R})$ transformations. As the latter are real, we are forced to discard $e^{i(\mathcal{L}_n+\mathcal{L}_n^\dagger)}$ and to use $e^{\mathcal{L}_n-\mathcal{L}_n^\dagger}$ only. This fact is strictly related to the nonabelian nature of the group $\pi_1(\Sigma)$.

Let us consider the operators

$$\mathcal{U}_k = e^{\lambda_k(\partial_k + iA_k)}, \qquad \lambda_k \in \mathbb{R}, \qquad k = 1, 2, \tag{7}$$

We also introduce the functions $F_k(x_1, x_2)$ defined by

$$\mathcal{U}_k = F_k e^{\lambda_k\partial_k} F_k^{-1}, \qquad k = 1, 2. \tag{8}$$

The identity $Af(B)A^{-1} = f(ABA^{-1})$ and Eq.(8) give $(\partial_k + iA_k)F_k = 0$. Also note that $e^{\lambda_k\partial_k}F_k^{-1}(\{x_k\}) = F_k^{-1}(\{x_j + \delta_{jk}\lambda_k\})e^{\lambda_k\partial_k}$. Therefore, defining $G_k(x_1, x_2)$ by

$$\mathcal{U}_k = G_k e^{\lambda_k\partial_k}, \qquad k = 1, 2, \tag{9}$$

we conclude that

$$F_k(\{x_j + \delta_{jk}\lambda_k\}) = G_k^{-1}(\{x_j\})F_k(\{x_j\}). \tag{10}$$

The unitary operators $\mathcal{U}_k$ can be used to derive the phase of Eq.(3). First we note that pulling the derivatives to the right we get

$$\begin{aligned}
\mathcal{U}_1\mathcal{U}_2\mathcal{U}_1^{-1}\mathcal{U}_2^{-1} &= F_1 e^{\lambda_1\partial_1}F_1^{-1}F_2 e^{\lambda_2\partial_2}F_2^{-1}F_1 e^{-\lambda_1\partial_1}F_1^{-1}F_2 e^{-\lambda_2\partial_2}F_2^{-1} \\
&= F_1(x_1,x_2)F_1^{-1}(x_1+\lambda_1,x_2)F_2(x_1+\lambda_1,x_2)F_2^{-1}(x_1+\lambda_1,x_2+\lambda_2) \\
&\times F_1(x_1+\lambda_1,x_2+\lambda_2)F_1^{-1}(x_1,x_2+\lambda_2)F_2(x_1,x_2+\lambda_2)F_2^{-1}(x_1,x_2).
\end{aligned} \tag{11}$$

Let us consider the curvature of $A = A_1dx_1 + A_2dx_2$

$$F = dA = (\partial_1 A_2 - \partial_2 A_1)dx_1 \wedge dx_2 = F_{12}dx_1 \wedge dx_2. \tag{12}$$

The constant–curvature connection is the unique possible choice to get a constant phase, so we set $F_{12} = 2\pi\theta$. To be explicit we pick the gauge $A_1 = -\pi\theta x_2$, $A_2 = \pi\theta x_1$, so that $\mathcal{U}_1 = e^{-i\pi\lambda_1\theta x_2}e^{\lambda_1\partial_1}$, $\mathcal{U}_2 = e^{i\pi\lambda_2\theta x_1}e^{\lambda_2\partial_2}$, and

$$G_1 = e^{-i\pi\lambda_1\theta x_2} = e^{i\lambda_1 A_1}, \qquad G_2 = e^{i\pi\lambda_2\theta x_1} = e^{i\lambda_2 A_2}, \tag{13}$$

so Eq.(10) reads

$$F_1(x_1+\lambda_1,x_2) = e^{i\pi\lambda_1\theta x_2}F_1(x_1,x_2), \qquad F_2(x_1,x_2+\lambda_2) = e^{-i\pi\lambda_2\theta x_1}F_2(x_1,x_2). \tag{14}$$

The solution is $F_1 = e^{i\pi\theta x_1 x_2}f_1(x_2)$, $F_2 = e^{-i\pi\theta x_1 x_2}f_2(x_1)$, with f_1 (f_2) an arbitrary function of x_2 (x_1). Substituting this into (11) we get Eq.(3), as we would using BCH.

From (9)(13) one would understand that the connection in (7) can be simply pulled to the left. However, this is the case only if one chooses a particular gauge, as in general we have $e^{\lambda_k(\partial_k+iA_k)} \neq e^{i\lambda_k A_k}e^{\lambda_k\partial_k}$. Indeed, under the gauge transformation $A_k \longrightarrow A_k + \partial_k\chi$, we have

$$e^{\lambda_k(\partial_k+iA_k+i\partial_k\chi)} = e^{-i\chi}e^{\lambda_k(\partial_k+iA_k)}e^{i\chi} = e^{i\chi(\{x_j+\delta_{jk}\lambda_j\})-i\chi(\{x_j\})}e^{\lambda_k(\partial_k+iA_k)}, \tag{15}$$

whereas under a gauge transformation, $e^{i\lambda_k A_k}e^{\lambda_k\partial_k}$ is multiplied by $e^{i\lambda_k\chi(\{x_j\})}$. It is easily seen that the correct expression is

$$e^{\lambda_k(\partial_k+iA_k)} = e^{i\int_{x_k}^{x_k+\lambda_k} da_k A_k}e^{\lambda_k\partial_k}, \tag{16}$$

where in the integrand one has $A_1(a_1,x_2)$ if $k=1$ and $A_2(x_1,a_2)$ if $k=2$. In (16) we used a shorthand notation; the integration limits should be written more precisely as $\int_{\{x_j\}}^{\{x_j+\delta_{jk}\lambda_j\}}$. In

particular, the contour is easily recognized as the path joining x_k and $x_k + \lambda_k$ along the line with $x_{j\neq k}$ fixed. Since on the torus we can choose the zero curvature metric, straight lines correspond to geodesics of the metric. Thus, the above contour is the geodesic joining $\{x_j\}$ with $\{x_j + \delta_{jk}\lambda_k\}$. A direct check of Eq.(16) is that

$$e^{\lambda_k(\partial_k + iA_k)} = e^{-i\int_{x_k^0}^{x_k} da_k A_k} e^{\lambda_k\partial_k} e^{i\int_{x_k^0}^{x_k} da_k A_k} = e^{i\int_{x_k}^{x_k+\lambda_k} da_k A_k} e^{\lambda_k\partial_k}, \tag{17}$$

where we used the property that $\partial_k \int_{x_{k0}}^{x_k} da_k A_k = A_k(x_1, x_2)$. This is a distinguished feature due to the flatness of the torus that does not hold in $g > 1$. However, we redefine the contour integral for the torus in a way which easily generalizes to higher genus, namely,

The contour integral is along the geodesic, with respect to the constant curvature metric, joining the points with coordinates $\{x_j\}$ *and* $\{x_j + \delta_{jk}\lambda_k\}$.

Due to the fact that along the integration contour either $dx_1 = 0$ or $dx_2 = 0$, we can replace $da_k A_k$ with A:

$$\mathcal{U}_k = e^{\lambda_k(\partial_k + iA_k)} = e^{-i\int_{x_k^0}^{x_k} A} e^{\lambda_k\partial_k} e^{i\int_{x_k^0}^{x_k} A} = e^{i\int_{x_k}^{x_k+\lambda_k} A} e^{\lambda_k\partial_k}, \tag{18}$$

so that $F_k = e^{-i\int_{x_k^0}^{x_k} A}$. Even if on the torus the F_k are not essential, we introduced them as their higher–genus analog will lead to a new class of functions. By Stokes' theorem

$$\mathcal{U}_1\mathcal{U}_2\mathcal{U}_1^{-1}\mathcal{U}_2^{-1} =$$

$$\exp\left[i\int_{(x_1,x_2)}^{(x_1+\lambda_1,x_2)} A + i\int_{(x_1+\lambda_1,x_2)}^{(x_1+\lambda_1,x_2+\lambda_2)} A + i\int_{(x_1+\lambda_1,x_2+\lambda_2)}^{(x_1,x_2+\lambda_2)} A + i\int_{(x_1,x_2+\lambda_2)}^{(x_1,x_2)} A\right]$$

$$= \exp\left(i\oint_{\partial\mathcal{F}} A\right) = \exp\left(i\int_{\mathcal{F}} F\right) = e^{2\pi i\lambda_1\lambda_2\theta}, \tag{19}$$

where $\mathcal{F}$ is a fundamental domain for the torus. Note that $\lambda_1\lambda_2$ is the area of the torus. Normalizing the area to 1, we get Eq.(3).

We now show that the only possible connection leading to a constant value of $\oint_{\partial\mathcal{F}} A$ is the one with constant curvature. In order to denote the dependence on the basepoint of the domain we use the notation $\mathcal{F}_{x_1x_2}$. Independence from (x_1, x_2) means that

$$\int_{\mathcal{F}_{x_1x_2}} F = \int_{\mathcal{F}_{x_1'x_2'}} F, \tag{20}$$

for any $(x_1', x_2') \in \mathbb{R}^2$. Any point in $\mathbb{R}^2$ can be obtained by a translation $(x_1, x_2) \to (x_1', x_2') = \mu(x_1, x_2) \equiv (x_1 + b_1, x_2 + b_2)$. Noticing that $\mathcal{F}_{x_1'x_2'} = \mu\mathcal{F}_{x_1x_2}$, we see that Eq.(20) is satisfied only if the curvature two–form F is invariant under arbitrary translations of (x_1, x_2). This fixes F to be a constant two–form.

The above investigation captures the essence of the construction in $g = 1$, somehow extracting it from its specific context. This is very useful to reformulate the problem of deriving a projective unitary representation of the fundamental group of a class of manifolds which is much more general than the torus. We can say that in order to get a projective unitary representation of the fundamental group of a given manifold $\mathcal{M}$ by means of operators acting on the space $L^2(\mathcal{M})$, we should consider the previous well–defined guidelines.

3. Projective unitary representation of $\pi_1(\Sigma)$ on $L^2(\mathbb{H})$.

We now apply the above general analysis to the case of higher genus Riemann surfaces. We start by first considering a unitary representation of $\pi_1(\Sigma)$ realized on $L^2(\mathbb{H})$ (the analog of $e^{\lambda_k \partial_k}$). For $n = -1, 0, 1$ and $e_n(z) = z^{n+1}$ let us set $\ell_n = e_n(z)\partial_z$. Define

$$L_n = e_n^{-1/2} \ell_n e_n^{1/2} = e_n \left(\partial_z + \frac{1}{2} \frac{e_n'}{e_n} \right). \tag{21}$$

They satisfy the $\mathrm{sl}_2(\mathbb{R})$ algebra

$$[L_m, L_n] = (n-m) L_{m+n}, \quad [\bar{L}_m, L_n] = 0, \quad [L_n, f] = z^{n+1} \partial_z f. \tag{22}$$

For $k = 1, 2, \ldots, 2g$, consider the operators

$$T_k = e^{\lambda_{-1}^{(k)}(L_{-1} + \bar{L}_{-1})} e^{\lambda_0^{(k)}(L_0 + \bar{L}_0)} e^{\lambda_1^{(k)}(L_1 + \bar{L}_1)}, \tag{23}$$

with the $\lambda_n^{(k)}$ picked such that

$$T_k z T_k^{-1} = \gamma_k z = \frac{a_k z + b_k}{c_k z + d_k}, \tag{24}$$

so that by (5)

$$\prod_{k=1}^{g} \left(T_{2k-1} T_{2k} T_{2k-1}^{-1} T_{2k}^{-1} \right) = \mathbb{I}. \tag{25}$$

On $L^2(\mathbb{H})$ we have the scalar product $\langle \phi | \psi \rangle = \int_{\mathbb{H}} d\nu \bar{\phi} \psi$, with $d\nu(z) = i dz \wedge d\bar{z}/2 = dx \wedge dy$. One can check that the T_k provide a unitary representation of Γ.

For any function F satisfying $|F| = 1$, we define the operators

$$\mathcal{L}_n^{(F)} = F(z, \bar{z}) L_n F^{-1}(z, \bar{z}) = e_n \left(\partial_z + \frac{1}{2} \frac{e_n'}{e_n} - \partial_z \ln F(z, \bar{z}) \right), \tag{26}$$

which also satisfy the algebra (22). Its adjoint is given by

$$\mathcal{L}_n^{(F)\dagger} = -F \overline{e_n^{1/2}} \partial_{\bar{z}} \overline{e_n^{1/2}} F^{-1} = -\bar{\mathcal{L}}_n^{(F^{-1})}. \tag{27}$$

We now observe that the operators

$$\Lambda_n^{(F)} = \mathcal{L}_n^{(F)} - \mathcal{L}_n^{(F)\dagger} = \mathcal{L}_n^{(F)} + \bar{\mathcal{L}}_n^{(F^{-1})}, \tag{28}$$

enjoy the fundamental property that both their chiral components are gauged in the same way by the function F, that is

$$\Lambda_n^{(F)} = F(L_n + \bar{L}_n)F^{-1}, \tag{29}$$

while also satisfying the $\mathrm{sl}_2(\mathbb{R})$ algebra:

$$[\Lambda_m^{(F)}, \Lambda_n^{(F)}] = (n-m)\Lambda_{m+n}^{(F)}, \qquad [\Lambda_n^{(F)}, f] = (z^{n+1}\partial_z + \bar{z}^{n+1}\partial_{\bar{z}})f. \tag{30}$$

Furthermore, since $\Lambda_n^{(F)\dagger} = -\Lambda_n^{(F)}$, the operators $e^{\Lambda_n^{(F)}} = Fe^{L_n+\bar{L}_n}F^{-1}$ are unitary.

Let b be a real number, and A a Hermitean connection to be identified presently. Set

$$\mathcal{U}_k = e^{ib\int_z^{\gamma_k z} A}T_k, \tag{31}$$

where the integration contour is taken to be the Poincaré geodesic connecting z and $\gamma_k z$. As the gauging functions introduced in (26) we will take the $F_k(z,\bar{z})$ solutions of the equation $F_kT_kF_k^{-1} = e^{ib\int_z^{\gamma_k z} A}T_k$, that is

$$F_k(\gamma_k z, \gamma_k \bar{z}) = e^{-ib\int_z^{\gamma_k z} A}F_k(z,\bar{z}). \tag{32}$$

With the choice (32) for F_k, (29) becomes

$$\Lambda_{n,k}^{(F)} = F_k(L_n + \bar{L}_n)F_k^{-1} = z^{n+1}\left(\partial_z + \frac{n+1}{2z} - \partial_z \ln F_k\right) + \bar{z}^{n+1}\left(\partial_{\bar{z}} + \frac{n+1}{2\bar{z}} - \partial_{\bar{z}} \ln F_k\right). \tag{33}$$

The $\Lambda_{n,k}^{(F)}$ satisfy the algebra

$$[\Lambda_{m,j}^{(F)}, \Lambda_{n,k}^{(F)}] = (n-m)\Lambda_{m+n,j}^{(F)} + F_k^{-1}|e_n|\Lambda_{n,k}^{(F)}|e_n|^{-1}F_kF_j^{-1}|e_m|\Lambda_{m,j}^{(F)}|e_m|^{-1}F_j(\ln F_j - \ln F_k),$$

$$[\Lambda_{n,k}^{(F)}, f] = (z^{n+1}\partial_z + \bar{z}^{n+1}\partial_{\bar{z}})f. \tag{34}$$

Upon exponentiating $\Lambda_{n,k}^{(F)}$ one finds

$$\mathcal{U}_k = e^{\lambda_{-1}^{(k)}\Lambda_{-1,k}^{(F)}}\, e^{\lambda_0^{(k)}\Lambda_{0,k}^{(F)}}\, e^{\lambda_1^{(k)}\Lambda_{1,k}^{(F)}}, \tag{35}$$

that is, the $\mathcal{U}_k$ are unitary, and

$$\mathcal{U}_k^{-1} = T_k^{-1}e^{-ib\int_z^{\gamma_k z} A} = e^{-ib\int_{\gamma_k^{-1} z}^{z} A}T_k^{-1}. \tag{36}$$

It is immediate to see that the $\mathcal{U}_k$ defined in (31) satisfy (6) for a certain value of θ:[1]

$$\prod_{k=1}^{g}\left(\mathcal{U}_{2k-1}\mathcal{U}_{2k}\mathcal{U}_{2k-1}^{\dagger}\mathcal{U}_{2k}^{\dagger}\right) = e^{ib\int_z^{\gamma_1 z}A}T_1 e^{ib\int_z^{\gamma_2 z}A}T_2 e^{-ib\int_{\gamma_1^{-1}z}^{z}A}T_1^{-1}e^{-ib\int_{\gamma_2^{-1}z}^{z}A}T_2^{-1}\ldots$$

$$= \exp\left[ib\left(\int_z^{\gamma_1 z}+\int_{\gamma_1 z}^{\gamma_2\gamma_1 z}+\int_{\gamma_2\gamma_1 z}^{\gamma_1^{-1}\gamma_2\gamma_1 z}+\int_{\gamma_1^{-1}\gamma_2\gamma_1 z}^{\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1 z}+\ldots\right)A\right]\prod_{k=1}^{g}\left(T_{2k-1}T_{2k}T_{2k-1}^{-1}T_{2k}^{-1}\right)$$

$$= e^{ib\oint_{\partial\mathcal{F}_z}A}, \tag{37}$$

where $\mathcal{F}_z = \{z, \gamma_1 z, \gamma_2\gamma_1 z, \gamma_1^{-1}\gamma_2\gamma_1 z, \ldots\}$ is a fundamental domain for Γ. The basepoint z, plus the action of the Fuchsian generators on it, determine $\mathcal{F}_z$, as the vertices are joined by geodesics.

For (37) to provide a projective unitary representation of Γ, $\int_{\mathcal{F}_z} dA$ should be z–independent. Changing z to z' can be expressed as $z \to z' = \mu z$ for some $\mu \in \mathrm{PSL}_2(\mathbb{R})$. Then $\mathcal{F}_z \to \mathcal{F}_{\mu z} = \{\mu z, \gamma_1\mu z, \gamma_2\gamma_1\mu z, \gamma_1^{-1}\gamma_2\gamma_1\mu z, \ldots\}$. Now consider $\mathcal{F}_z \to \mu\mathcal{F}_z = \{\mu z, \mu\gamma_1 z, \mu\gamma_2\gamma_1 z, \mu\gamma_1^{-1}\gamma_2\gamma_1 z, \ldots\}$. The congruence $\mu\mathcal{F}_z \cong \mathcal{F}_{\mu z}$ follows from two facts: that the vertices are joined by geodesics, and that $\mathrm{PSL}_2(\mathbb{R})$ maps geodesics into geodesics. Since Γ is defined up to conjugation, $\Gamma \to \mu\Gamma\mu^{-1}$, if $\mu\mathcal{F}_z$ is a fundamental domain, so is $\mathcal{F}_{\mu z}$. Thus, to have z–independence we need $\forall\mu \in \mathrm{PSL}_2(\mathbb{R})$

$$\int_{\mathcal{F}_z} dA = \int_{\mathcal{F}_{\mu z}} dA = \int_{\mu\mathcal{F}_z} dA = \int_{\mathcal{F}} dA. \tag{38}$$

This fixes the (1,1)–form dA to be $\mathrm{PSL}_2(\mathbb{R})$–invariant. It is well known that the Poincaré form is the unique $\mathrm{PSL}_2(\mathbb{R})$–invariant (1,1)–form, up to an overall constant factor. This is a particular case of a more general fact [20]. The Poincaré metric $ds^2 = y^{-2}|dz|^2 = 2g_{z\bar{z}}|dz|^2 = e^{\varphi}|dz|^2$ has curvature $R = -g^{z\bar{z}}\partial_z\partial_{\bar{z}}\ln g_{z\bar{z}} = -1$, so that $\int_{\mathcal{F}} d\nu e^{\varphi} = -2\pi\chi(\Sigma)$, where $\chi(\Sigma) = 2-2g$ is the Euler characteristic. As the Poincaré (1,1)–form is $dA = e^{\varphi}d\nu$, this uniquely determines the gauge field to be

$$A = A_z dz + A_{\bar{z}} d\bar{z} = \frac{dx}{y}, \tag{39}$$

modulo gauge transformations. Using $\oint_{\partial\mathcal{F}} A = \int_{\mathcal{F}} dA$ we finally have that (37) becomes

$$\prod_{k=1}^{g}\left(\mathcal{U}_{2k-1}\mathcal{U}_{2k}\mathcal{U}_{2k-1}^{\dagger}\mathcal{U}_{2k}^{\dagger}\right) = e^{2\pi ib\chi(\Sigma)}. \tag{40}$$

4. Nonabelian gauge fields.

Up to now we considered the case in which the connection is Abelian. However, it is easy to extend our construction to the nonabelian case in which the gauge group $U(1)$ is replaced by $U(N)$. The operators $\mathcal{U}_k$ now become

$$\mathcal{U}_k = Pe^{ib\int_z^{\gamma_k z}A}T_k, \tag{41}$$

[1] The differential representation of $\mathrm{PSL}_2(\mathbb{R})$ acts in reverse order with respect to the one by matrices.

where the T_k are the same as before, times the $N \times N$ identity matrix. Eq.(37) is replaced by

$$\prod_{k=1}^{g} \left(\mathcal{U}_{2k-1} \mathcal{U}_{2k} \mathcal{U}_{2k-1}^{\dagger} \mathcal{U}_{2k}^{\dagger} \right) = P e^{ib \oint_{\partial \mathcal{F}_z} A}. \tag{42}$$

Given an integral along a closed contour σ_z with basepoint z, the path–ordered exponentials for a connection A and its gauge transform $A^U = U^{-1}AU + U^{-1}dU$ are related by [21]

$$Pe^{i \oint_{\sigma_z} A} = U(z) P e^{i \oint_{\sigma_z} A^U} U^{-1}(z) = U(z) P e^{i \oint_{\sigma_z} d\sigma^\mu \int_0^1 dss\sigma^\nu U^{-1}(s\sigma) F_{\nu\mu}(s\sigma) U(s\sigma)} U^{-1}(z). \tag{43}$$

Applying this to (42), we see that the only possibility to get a coordinate–independent phase is for the curvature (1,1)–form $F = dA + [A, A]/2$ to be the identity matrix in the gauge indices times a (1,1)–form η, that is $F = \eta \mathbb{I}$. It follows that

$$P e^{ib \oint_{\partial \mathcal{F}} A} = e^{ib \int_{\mathcal{F}} F}. \tag{44}$$

This is only a necessary condition for coordinate–independence. However, this is the same as the Abelian case so that η should be proportional to the Poincaré (1,1)–form.

Denoting by E the vector bundle on which A is defined, we have $k = \deg(E) = \frac{1}{2\pi} \mathrm{tr} \int_{\mathcal{F}} F$. Set $\mu(E) = k/N$ so that $\int_{\mathcal{F}} F = 2\pi \mu(E) \mathbb{I}$ and $\eta = -\frac{\mu(E)}{\chi(\Sigma)} e^{\varphi} d\nu$, *i.e.*

$$F = 2\pi \mu(E) \omega \mathbb{I}, \tag{45}$$

where $\omega = \left(e^{\varphi} / \int_{\mathcal{F}} d\nu e^{\varphi}\right) d\nu$. Thus, by (44) we have that Eq.(42) becomes

$$\prod_{k=1}^{g} \left(\mathcal{U}_{2k-1} \mathcal{U}_{2k} \mathcal{U}_{2k-1}^{\dagger} \mathcal{U}_{2k}^{\dagger} \right) = e^{2\pi i b \mu(E)} \mathbb{I}, \tag{46}$$

which provides a projective unitary representation of $\pi_1(\Sigma)$ on $L^2(\mathbb{H}, \mathbb{C}^N)$.

5. Hochschild cohomology and gauge lengths.

A basic object is the *gauge length* function $d_A(z, w) = \int_z^w A$, where the contour integral is along the Poincaré geodesic connecting z and w. In the Abelian case

$$d_A(z, w) = \int_{\mathrm{Re}\, z}^{\mathrm{Re}\, w} \frac{dx}{y} = -i \ln \left(\frac{z - \bar{w}}{w - \bar{z}} \right), \tag{47}$$

which is equal to the angle α_{zw} spanned by the arc of geodesic connecting z and w. Observe that the gauge length of the geodesic connecting two punctures, *i.e.* two points on the real line, is π. This is to be compared with the usual divergence of the Poincaré distance. Under a $\mathrm{PSL}_2(\mathbb{R})$–transformation μ, we have ($\mu_x \equiv \partial_x \mu x$)

$$d_A(\mu z, \mu w) = d_A(z, w) - \frac{i}{2} \ln \left(\frac{\mu_z \bar{\mu}_w}{\bar{\mu}_z \mu_w} \right). \tag{48}$$

Therefore, the gauge length of an n–gon

$$d_A^{(n)}(\{z_k\}) = \sum_{k=1}^{n} d_A(z_k, z_{k+1}) = \pi(n-2) - \sum_{k=1}^{n} \alpha_k, \tag{49}$$

where $z_{n+1} \equiv z_1$, $n \geq 3$, and α_k are the internal angles, is $\mathrm{PSL}_2(\mathbb{R})$–invariant.

We now show that the length of the triangle is proportional to the Hochschild 2–cocycle of Γ. The Fuchsian generators $\gamma_k \in \Gamma$ are projectively represented by means of unitary operators $\mathcal{U}_k$ acting on $L^2(\mathbb{H})$. The product $\gamma_k\gamma_j$ is represented by $\mathcal{U}_{jk}$, which equals $\mathcal{U}_j\mathcal{U}_k$ up to a phase:

$$\mathcal{U}_j\mathcal{U}_k = e^{2\pi i\theta(j,k)}\mathcal{U}_{jk}. \tag{50}$$

Associativity implies

$$\theta(j,k) + \theta(jk,l) = \theta(j,kl) + \theta(k,l). \tag{51}$$

We can easily determine $\theta(j,k)$:

$$\mathcal{U}_j\mathcal{U}_k = \exp\left(ib\int_z^{\gamma_j z} A + ib\int_{\gamma_j z}^{\gamma_k\gamma_j z} A - ib\int_z^{\gamma_k\gamma_j z} A\right)\mathcal{U}_{jk} = \exp\left(ib\int_{\tau_{jk}} A\right)\mathcal{U}_{jk}, \tag{52}$$

where τ_{jk} denotes the geodesic triangle with vertices z, $\gamma_j z$ and $\gamma_k\gamma_j z$. This identifies $\theta(j,k)$ as the gauge length of the perimeter of the geodesic triangle τ_{jk} times $b/2\pi$. By Stokes' theorem this is the Poincaré area of the triangle. One can check that $\theta(j,k)$ in fact satisfies (51). This phase has been considered in different contexts, such as the quantum Hall effect on $\mathbb{H}$ [22] and Berezin's quantization of $\mathbb{H}$ and Von Neumann algebras [23].

The information on the compactification of M(atrix) theory is encoded in the action of Γ on $\mathbb{H}$, plus a projective representation of Γ. The latter amounts to the choice of a phase. Physically inequivalent choices of $\theta(j,k)$ turn out to be in one–to–one correspondence with elements in the 2nd Hochschild cohomology group of Γ, which is $U(1)$. Hence $\theta = b\chi(\Sigma)$ is the unique parameter for this compactification ($\theta = b\mu(E)$ in the general case).

The Poincaré metric is $\mathrm{PSL}_2(\mathbb{R})$ invariant whereas A is not. So the equality $\oint_{\partial\mathcal{F}} A = \int_{\mathcal{F}} F$ should be a consequence of the fact that the variation of A under a $\mathrm{PSL}_2(\mathbb{R})$ transformation, $z \longrightarrow \mu z = (az+b)/(cz+d)$, corresponds to a total derivative. In fact we have

$$\mathrm{PSL}_2(\mathbb{R}) : A \longrightarrow i\frac{d\mu z + d\mu\bar{z}}{\mu z - \mu\bar{z}} = A - i\partial_z \ln(cz+d)dz + i\partial_{\bar{z}} \ln(c\bar{z}+d)d\bar{z}. \tag{53}$$

Since $cz+d$ has no zeroes, we have that $\ln(cz+d)$ is a genuine function on $\mathbb{H}$. It follows that $-i\partial_z \ln(cz+d)dz + i\partial_{\bar{z}} \ln(c\bar{z}+d)d\bar{z}$, can be written as an external derivative so that Eq.(53) becomes

$$\mathrm{PSL}_2(\mathbb{R}) : A \longrightarrow A + d\ln(\mu_z/\bar{\mu}_z)^{\frac{i}{2}}, \tag{54}$$

where $\mu_z \equiv \partial_z \mu z$. So a $\mathrm{PSL}_2(\mathbb{R})$–transformation of A is equivalent to a gauge transformation. Under $A \rightarrow A + d\chi$ we have $\int_z^w A \longrightarrow \int_z^w A + \chi(w) - \chi(z)$, which for $\chi(z) = \ln(\mu_z/\bar{\mu}_z)^{\frac{i}{2}}$, becomes

$$\int_z^w A \longrightarrow \int_z^w A + \frac{i}{2} \ln \frac{\bar{\mu}_z \mu_w}{\mu_z \bar{\mu}_w}. \tag{55}$$

6. Preautomorphic forms.

Another reason why the gauge–length function is important is that it also appears in the definition (32) of the F_k. The latter functions, which apparently never appeared in the literature before, are of particular interest. By (32) and (47),

$$F_k(\gamma_k z, \gamma_k \bar{z}) = \left(\frac{\gamma_k z - \bar{z}}{z - \gamma_k \bar{z}} \right)^b F_k(z, \bar{z}). \tag{56}$$

Since under a $\mathrm{PSL}_2(\mathbb{R})$ transformation the factor $(w - \bar{z})/(z - \bar{w})$ gets transformed by a factor which is typical of automorphic forms, we call the F_k *preautomorphic forms.* Eq.(32) indicates that finding the most general solution to (56) is a problem in geodesic analysis. In the case of the inversion $\gamma_k z = -1/z$ and b an even integer, a solution to (56) is $F_k = (z/\bar{z})^{\frac{b}{2}}$. By (47) $F_k = (z/\bar{z})^{\frac{b}{2}}$ is related to the A*–length* of the geodesic connecting z and 0:

$$e^{\frac{i}{2} b \int_z^0 A} = F_k(z, \bar{z}) = \left(\frac{z}{\bar{z}} \right)^{\frac{b}{2}}. \tag{57}$$

An interesting formal solution to (56) is

$$F_k(z, \bar{z}) = \prod_{j=0}^{\infty} \left(\frac{\gamma_k^{-j} z - \gamma_k^{-j-1} \bar{z}}{\gamma_k^{-j-1} z - \gamma_k^{-j} \bar{z}} \right)^b. \tag{58}$$

Consider the uniformizing map $J_{\mathbb{H}} : \mathbb{H} \longrightarrow \Sigma$, which enjoys the property $J_{\mathbb{H}}(\gamma z) = J_{\mathbb{H}}(z)$, $\forall \gamma \in \Gamma$. Then another solution to (56) is given by $G(J_{\mathbb{H}}, \bar{J}_{\mathbb{H}}) F_k$, where G is an arbitrary function of the uniformizing map. We should require $|G| = 1$ for $|F_k| = 1$.

7. Relation with Donaldson's approach to stable bundles.

We now present some facts about projective, unitary representations of Γ and the theory of holomorphic vector bundles [24, 25]. Let $E \rightarrow \Sigma$ be a holomorphic vector bundle over Σ of rank N and degree k. The bundle E is called *stable* if the inequality $\mu(E') < \mu(E)$ holds for every proper holomorphic subbundle $E' \subset E$. We may take $-N < k \leq 0$. We will further assume that Γ contains a unique primitive elliptic element γ_0 of order N (*i.e.*, $\gamma_0^N = \mathbb{I}$), with fixed point $z_0 \in \mathbb{H}$ that projects to $x_0 \in \Sigma$.

Given the branching order N of γ_0, let $\rho : \Gamma \to U(N)$ be an irreducible unitary representation. It is said *admissible* if $\rho(\gamma_0) = e^{-2\pi i k/N}\mathbb{I}$. Putting the elliptic element on the right–hand side, and setting $\rho_k \equiv \rho(\gamma_k)$, (5) becomes

$$\prod_{j=1}^{g} \left(\rho_{2j-1}\rho_{2j}\rho_{2j-1}^{-1}\rho_{2j}^{-1}\right) = e^{2\pi i k/N}\mathbb{I}. \tag{59}$$

On the trivial bundle $\mathbb{H} \times \mathbb{C}^N \to \mathbb{H}$ there is an action of Γ: $(z, v) \to (\gamma z, \rho(\gamma)v)$. This defines the quotient bundle

$$\mathbb{H} \times \mathbb{C}^N/\Gamma \to \mathbb{H}/\Gamma \cong \Sigma. \tag{60}$$

Any admissible representation determines a holomorphic vector bundle $E_\rho \to \Sigma$ of rank N and degree k. When $k = 0$, E_ρ is simply the quotient bundle (60) of $\mathbb{H} \times \mathbb{C}^N \to \mathbb{H}$. The Narasimhan–Seshadri (NS) theorem [26] now states that a holomorphic vector bundle E over Σ of rank N and degree k is stable if and only if it is isomorphic to a bundle E_ρ, where ρ is an admissible representation of Γ. Moreover, the bundles E_{ρ_1} and E_{ρ_2} are isomorphic if and only if the representations ρ_1 and ρ_2 are equivalent.

A differential–geometric approach to stability has been given by Donaldson [27]. Fix a Hermitean metric on Σ, for example the Poincaré metric, normalized so that the area of Σ equals 1. Let us denote by ω its associated (1,1)–form. A holomorphic bundle E is stable if and only if there exists on E a metric connection A_D with central curvature $F_D = -2\pi i\mu(E)\omega\mathbb{I}$; such a connection A_D is unique.

The unitary projective representations of Γ we constructed above have a uniquely defined gauge field whose curvature is proportional to the volume form on Σ. With respect to the representation considered by NS, we note that NS introduced an elliptic point to produce the phase, while in our case the latter arises from the gauge length. Our construction is directly connected with Donaldson's approach as $F = iF_D$, where F is the curvature (45). The main difference is that our operators are unitary differential operators on $L^2(\mathbb{H}, \mathbb{C}^N)$ instead of unitary matrices on $\mathbb{C}^N$. This allowed us to obtain a non–trivial phase also in the Abelian case.

It is however possible to understand the formal relation between our operators and those of NS. To see this we consider the adjoint representation of Γ on $\operatorname{End}\mathbb{C}^N$,

$$\operatorname{Ad}\rho(\gamma)Z = \rho(\gamma)Z\rho^{-1}(\gamma), \tag{61}$$

where $Z \in \operatorname{End}\mathbb{C}^N$ is understood as an $N \times N$ matrix. Let us also consider the trivial bundle $\mathbb{H} \times \operatorname{End}\mathbb{C}^N \to \mathbb{H}$. The action of Γ $(z, Z) \mapsto (\gamma z, \operatorname{Ad}\rho(\gamma)Z)$ defines the quotient bundle

$$\mathbb{H} \times \operatorname{End}\mathbb{C}^N/\Gamma \to \mathbb{H}/\Gamma \cong \Sigma. \tag{62}$$

Then the idea is to consider a vector bundle E' in the double scaling limit $N' \to \infty$, $k' \to -\infty$, with $\mu(E') = k'/N'$ fixed, that is $\mu(E') = b\mu(E)$. In this limit, fixing a basis in $L^2(\mathbb{H}, \mathbb{C}^N)$, the matrix elements of our operators can be identified with those of $\rho(\gamma)$.

8. Noncommutative uniformization.

Let us now introduce two copies of the upper half–plane, one with coordinates z and $\bar{z}$, the other with coordinates w and $\bar{w}$. While the coordinates z and $\bar{z}$ are reserved to the operators $\mathcal{U}_k$ we introduced previously, we reserve w and $\bar{w}$ to construct a new set of operators. We now introduce noncommutative coordinates expressed in terms of the covariant derivatives

$$W = \partial_w + iA_w, \qquad \bar{W} = \partial_{\bar{w}} + iA_{\bar{w}}, \tag{63}$$

with $A_w = A_{\bar{w}} = 1/(2\,\mathrm{Im}\,w)$, so that $[W, \bar{W}] = iF_{w\bar{w}}$, where $F_{w\bar{w}} = i/[2(\mathrm{Im}\,w)^2]$. Let us consider the following realization of the $\mathrm{sl}_2(\mathbb{R})$ algebra:

$$\hat{L}_{-1} = -w, \qquad \hat{L}_0 = -\frac{1}{2}(w\partial_w + \partial_w w), \qquad \hat{L}_1 = -\partial_w w \partial_w. \tag{64}$$

We then define the unitary operators

$$\hat{T}_k = e^{\lambda_{-1}^{(k)}(\hat{L}_{-1}+\bar{\hat{L}}_{-1})} e^{\lambda_0^{(k)}(\hat{L}_0+\bar{\hat{L}}_0)} e^{\lambda_1^{(k)}(\hat{L}_1+\bar{\hat{L}}_1)}, \tag{65}$$

where the $\lambda_n^{(k)}$ are as in (23). Set $\mathcal{V}_k = \hat{T}_k \mathcal{U}_k$. Since the $\hat{T}_k$ satisfy (25), it follows that the $\mathcal{V}_k$ satisfy (46), times the $N \times N$ identity matrix, and

$$\mathcal{V}_k \partial_w \mathcal{V}_k^{-1} = \hat{T}_k \partial_w \hat{T}_k^{-1} = \frac{a_k \partial_w + b_k}{c_k \partial_w + d_k}. \tag{66}$$

Setting $W = G\partial_w G^{-1}$, *i.e.* $G = (w - \bar{w})^2$, and using $Af(B)A^{-1} = f(ABA^{-1})$, we see that

$$\mathcal{V}_k W \mathcal{V}_k^{-1} = \hat{T}_k W \hat{T}_k^{-1} = G(\tilde{w}) \hat{T}_k \partial_w \hat{T}_k^{-1} G^{-1}(\tilde{w}), \tag{67}$$

where

$$\tilde{w} = \hat{T}_k w \hat{T}_k^{-1} = -e^{-\lambda_0^{(k)}} + 2\lambda_1^{(k)}(\hat{L}_0 - \lambda_{-1}^{(k)} w) - \lambda_1^{(k)2} e^{\lambda_0^{(k)}} (\hat{L}_1 + 2\lambda_{-1}^{(k)} \hat{L}_0 - \lambda_{-1}^{(k)2} w), \tag{68}$$

and by (66)

$$\mathcal{V}_k W \mathcal{V}_k^{-1} = \hat{T}_k W \hat{T}_k^{-1} = \frac{a_k \tilde{W} + b_k}{c_k \tilde{W} + d_k}, \tag{69}$$

where

$$\tilde{W} = \partial_w + G(\tilde{w})[\partial_w G^{-1}(\tilde{w})], \tag{70}$$

which differs from W by the connection term. Eq.(69) can be seen as representing the noncommutative analog of uniformization.

9. $C^\star$–algebra.

By a natural generalization of the n–dimensional noncommutative torus, one defines a noncommutative Riemann surface Σ_θ in $g>1$ to be an associative algebra with involution having unitary generators $\mathcal{U}_k$ obeying the relation (40). Such an algebra is a $C^\star$–algebra, as it admits a faithful unitary representation on $L^2(\mathbb{H},\mathbb{C}^N)$ whose image is norm–closed. Relation (40) is also satisfied by the $\mathcal{V}_k$. However, while the $\mathcal{U}_k$ act on the commuting coordinates $z,\bar{z}$, the $\mathcal{V}_k$ act on the operators W and $\bar{W}$. The latter, factorized by the action of the $\mathcal{V}_k$ in (69), can be pictorially identified with a sort of noncommutative coordinates on Σ_θ.

Each $\gamma\neq\mathbb{I}$ in Γ can be uniquely expressed as a positive power of a primitive element $p\in\Gamma$, *primitive* meaning that p is not a positive power of any other $p'\in\Gamma$ [28]. Let $\mathcal{V}_p$ be the representative of p. Any $\mathcal{V}\in C^\star$ can be written as

$$\mathcal{V}=\sum_{p\in\{prim\}}\sum_{n=0}^{\infty}c_n^{(p)}\mathcal{V}_p^n+c_0\mathbb{I}, \tag{71}$$

for certain coefficients $c_n^{(p)}$, c_0. A trace can be defined as $\mathrm{tr}\,\mathcal{V}=c_0$.

In the case of the torus one can connect the $C^\star$–algebras of $U(1)$ and $U(N)$. To see this one can use 't Hooft's clock and shift matrices V_1, V_2, which satisfy $V_1V_2=e^{2\pi i\frac{M}{N}}V_2V_1$. The $U(N)$ $C^\star$–algebra is constructed in terms of the V_k and of the unitary operators representing the $U(1)$ $C^\star$–algebra. Morita equivalence is an isomorphism between the two. In higher genus, the analog of the V_k is the $U(N)$ representation $\rho(\gamma)$ considered above. One can obtain a $U(N)$ projective unitary differential representation of Γ by taking $\mathcal{V}_k\rho(\gamma_k)$, with $\mathcal{V}_k$ Abelian. This nonabelian representation should be compared with the one obtained by the nonabelian $\mathcal{V}_k$ constructed above. In this framework it should be possible to understand a possible higher–genus analog of the Morita equivalence.

The isomorphism of the $C^\star$–algebras is a direct consequence of an underlying equivalence between the $U(1)$ and $U(N)$ connection. The z–independence of the phase requires F to be the identity matrix in the gauge indices. This in turn is deeply related to the uniqueness of the connection we found. The latter is related to the uniqueness of the NS connection. We conclude that Morita equivalence in higher genus is intimately related to the NS theorem.

Our operators correspond to the $N\to\infty$ limit of projective unitary representations of Γ. These operators may be useful in studying the moduli space of M(atrix) string theory [29]. They also play a role in the $N\to\infty$ limit of QCD as considered in [30].

Finally, let us note that an alternative proposal of noncommutative Riemann surfaces and $C^\star$–algebras has been considered in [31][22].

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